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Topology and its Applications 125 (2002) 343–355

TOPOLOGY AND ITS APPLICATIONS

www.elsevier.com/locate/topol

A Ramsey-type topological theorem ☆

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Received 24 June 2001; received in revised form 1 November 2001

Abstract

Generalizing a theorem of S. Todorčević and W. Weiss we prove that if X is a monotonically normal, first countable and not left separated space, then colouring the r-tuples of X with finitely many colours, there is a not discrete homogeneous subset of X.

We study also a certain weakening of left separability and some other properties related to the subject.

MSC: primary 54A25, 54D15, 03E05; secondary 54A35

Keywords: Partition calculus; Left separated space; Monotone normality; Stationary sets; Topological cardinal function; Colouring the r-tuples of a topological space

1. Preliminaries

A topological space is *left separated* if there is a well-order on X and a neighbourhood-assignment $x \to U(x)$ $(x \in X)$ such that $U(x)$ is a neighbourhood of the point x and $\min U(x) = x$ for each $x \in X$.

Theorem 1.1. *If X is regular and left separated, we can colour the pairs of X with* 2 *colours in such a way that each homogeneous subset of X is discrete.*

Proof. Choose a well-order $\prec$ and a neighbourhood-assignment $U(x)$ $(x \in X)$ witnessing that X is left separated. By regularity, we can assume that $U(x)$ is a closed neighbourhood of the point x for each $x \in X$. Put

$$c(x, y) = \begin{cases} 0 & \text{if } y \notin U(x), \\ 1 & \text{if } y \in U(x) \end{cases}$$

☆ Research partially supported by Hungarian National Foundation for Scientific Research grant no. 25745.

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0166-8641/01/$ – see front matter © 2001 Published by Elsevier Science B.V.
PII: S0166-8641(01)00287-5

for $x, y \in X$, $x \prec y$.

If $H \subset X$ is 0-homogeneous (the colour of any pair in H is 0) then $x \notin U(y)$ for any pair $x, y \in H$ hence H is evidently discrete. Assume now that $H \subset X$ is 1-homogeneous and take a point $p \in H$. Let q denote the minimal point in H which is greater than p. Now, by the 1-homogeneity of H, the neighborhood $U(p) - U(q)$ of the point p does not contain any other point of H. (If p is the greatest point in H then $U(p) \cap H = \emptyset$.) $\square$

The basic problem of our paper: *Is the converse to Theorem* 1.1 *true*? A recent result of S. Todorčević and W. Weiss states that in the class of metrizable spaces the answer is yes:

Theorem A (Todorčević, Weiss [7]). *If the r-tuples of a not left separated metrizable space are coloured with finitely many colours then there is a homogeneous convergent sequence* (*together with the limit point*).

Mimicking the notation of the partition calculus, the theorem says: *In the realm of metric spaces for any* $2 \leqslant r,\ n < \omega$

$$\mathrm{not\,LS} \to (\mathrm{top}\,\omega + 1)^r_n.$$

The main theorem of the paper generalizes Theorem A for monotonically normal first countable spaces.

A T_1-space X is called *monotonically normal* if we can assign to any pair (x, U), where $x \in X$ and U is an open set containing x, an open set U' which also contains the point x and the following property holds: for two pairs $(x, U), (y, V)$ if $x \notin V$ and $y \notin U$ then $U' \cap V' = \emptyset$.

Monotone normality is hereditary to subspaces. Any monotonically normal space is (hereditarily) normal. Metric spaces and order-topological spaces are monotonically normal [5].

The main result of the paper is the following:

Theorem 1.2. *For the natural numbers $2 \leqslant r,\ n < \omega$ if the r-tuples of a not left separated monotonically normal and first countable space X are coloured with n colours then there is a homogeneous convergent sequence*: $X \to (\mathrm{top}\,\omega + 1)^r_n$.

We give two proofs of this theorem in the paper. The proofs are based on two characterizations of the left separated spaces in the class of monotonically normal spaces. Both characterizations use the basic properties of stationary sets, but in two different senses: the proof given in Section 3 (which follows the method used by S. Todorčević and W. Weiss) uses the classical theory of stationary sets in regular cardinals, while the characterization of Section 2 uses the notion of the stationary sets in the family of countable subsets of a set developed by T. Jech.

Remark that for $r = 2$ S. Todorčević and W. Weiss state a stronger theorem:

Theorem B (Todorčević, Weiss). *If the pairs of a not left separated metrizable space are coloured with* 2 *colours then either there is a* 0*-homogeneous convergent sequence* (*together with the limit point*) *or there is a* 1*-homogeneous not left separated subset*; *i.e.,*

$$\mathrm{not\,LS} \to (\mathrm{top}\,\omega + 1, \mathrm{not\,LS}).$$

The following example shows that the proposition corresponding to Theorem B is not true in the larger class.

Example 1.3. Let S denote the Sorgenfrey line: the basic set is the set of the real numbers and the neighbourhood-base of a point $x \in S$ are the half-closed intervals $[x, y)$, where $x < y,\ x, y \in S$. As the Sorgenfrey line is a subspace of an ordered space (see, e.g., [2]), the space is monotonically normal. Apply the Sierpinski colouring to S: that is, choose a well-order $\prec$ on S and let the colour of a pair be 0 if their order is the same in the two orderings and 1 otherwise.

Observe now that does not exist a 0-homogeneous convergent sequence: this would mean a monotonically increasing convergent sequence in S. On the other hand, any 1-homogeneous set is countable hence left separated.

We state here two lemmas which are needed in the sequel. Probably both are known but we did not succeed to find them in the literature.

Lemma 1.4. *For a monotonically normal space the density and the hereditary density are identical.*

Proof. Let X be monotonically normal, $d(X) = \kappa$ and assume that there exist a subset $T \subset X$ with $d(T) > \kappa$. Then there is a left separated subset S with $|S| = \kappa^+$. Choose for each $x \in S$ an open neighbourhood $U(x)$ of x with $\min \overline{U(x)} \cap S = x$. Let $U'(x)$ denote the open set assigned to the pair $(x, U(x))$ by the definition of monotone normality; $U'(x) \subset U(x)$ can be assumed. As $d(X) = \kappa$, there is a subset $R \subset S$, $|R| = \kappa^+$ such that $\bigcap\{U'(x)\colon\ x \in R\} \neq \emptyset$. Hence, for $x, y \in R,\ x \in U(y)$ or $y \in U(x)$ holds. If $x \prec y$ in the well-order of R then $x \in U(y)$ is impossible, so $x, y \in R,\ x \prec y$ implies $y \in U(x)$. For $x \in R$ let x^+ denote the minimal point in R which is greater than x. Then $V(x) = U(x) - \overline{U(x^+)}$ is an open neighbourhood of x and $x \notin V(y)$ for $x, y \in R$, $x \neq y$.

Hence the open sets $V'(x)$ $(x \in R)$ are pairwise disjoint, contrary to the hypothesis $d(X) = \kappa$. $\square$

Lemma 1.5. *Let $\kappa > \omega$ be a regular cardinal, $S \subset \kappa$ a stationary subset in κ. If the pairs of S are coloured with* 2 *colours then either there is a* 0*-homogeneous set of type $\omega + 1$ or there is a* 1*-homogeneous stationary set $T \subset S$.*

Proof. Let $c(\xi, \eta) \in 2$ denote the colour assigned to the pair $\xi, \eta \in S$. Fix an ordinal $\xi < \kappa$ and try to choose a sequence $\langle \xi_i \colon i < \omega \rangle$ with the following properties: $\xi_i < \xi_j < \xi$ and $c(\xi_i, \xi_j) = c(\xi_i, \xi) = 0$ for $i < j < \omega$.

As an infinite sequence of this kind would give a 0-homogeneous set of type $\omega + 1$, we can assume that the process breaks down at a given point: we can construct the sequence $\langle \xi_i \colon i < k \rangle$ but a suitable ξ_k does not exist. (Of course, the number k and the sequence $\langle \xi_i \colon i < k \rangle$ depends on the ordinal ξ.) Take the regressive function $\xi \to \langle \xi_i \colon i < k \rangle$. By Fodor's theorem, there is a stationary set $T \subset S$ such that the number k and the assigned sequence is the same for each $\xi \in T$. As no sequence can be continued, we get that if $\xi, \eta \in T,\ \xi \prec \eta$ then $c(\xi, \eta) \neq 0$. □

2. Proof of the main theorem

The following property is intimately connected with our subject.

Definition. A space X is said to be *weakly left separated* if there is a neighbourhood-assignment U_x $(x \in X)$ such that for any infinite subset $A \subset X$ there is an infinite sequence $\{x_n\} \subset A$ with $x_n \notin U(x_m)$ for $n < m$.

Equivalently: for any infinite $A \subset X$ there is a point $x \in A$ such that the set $\{y \in A \colon x \notin U(y)\}$ is infinite.

Remark. It is an exercise to prove that the similar property "weakly right separated" is not interesting.

Indeed, *a space is weakly right separated iff right separated iff scattered*.

Theorem 2.1. *If X is a regular weakly left separated space then there is a colouring of the pairs of X with* 3 (!) *colours such that any homogeneous subset is discrete.*

Proof. Choose a neighbourhood-assignment $U(x)\,(x \in X)$ of X, witnessing that X is weakly left separated. We can suppose that $U(x)$ is a closed neighbourhood of x for any $x \in X$. Let $\prec$ be a well-order on X and put

$$c(x, y) = \begin{cases} 0 & \text{if } x \in U(y), \\ 1 & \text{if } x \notin U(y) \text{ and } y \in U(x), \\ 2 & \text{if } x \notin U(y) \text{ and } y \notin U(x), \end{cases}$$

for $x, y \in X$, $x \prec y$.

Assume that H is a homogeneous subset of X; we show that it is discrete. If H is 0-homogeneous then it is finite because X is weakly left separated. A 2-homogeneous set is evidently discrete so assume that H is 1-homogeneous and $p \in H$ is a limit-point of H. Let q denote the minimal point in H which is greater than p. Now, by the 1-homogeneity of H, the neighborhood $U(p) - U(q)$ of the point p does not contain any other point of H. (If p is the greatest point in H then $U(p) \cap H = \emptyset$.) □

Let $T \subset [X]^\omega$. We say that T is *unbounded* if for all $M \in [X]^\omega$ there is an $N \in T$ with $M \subset N$. T is *closed* if whenever $\{M_n\colon n \in \omega\}$ is an increasing chain from T then $\bigcup\{M_n\colon n \in \omega\} \in T$. T is a *club* if it is unbounded and closed. $S \subset [X]^\omega$ is *stationary* if it meets any club. A function $f : T \to X$ is called *regressive* if $f(M) \in M$ for all $M \in T$.

Fodor's theorem (actually proved by Jech [6]) states that if $|X| > \omega$, S is stationary in $[X]^\omega$ and f is regressive on S then there is a stationary $T \subset S$ and a point $a \in X$ such that $f(M) = a$ for each $M \in T$.

If $f : [X]^{<\omega} \to X$ and the image of any finite subset of the set $Y \subset X$ is an element of Y, we say that Y *is f-closed*. An important lemma of Kueker [1] states: if $C \subset [X]^\omega$ is a club then there is a function $f : [X]^{<\omega} \to X$ such that

$$\big\{M \in [X]^\omega\colon M \text{ is } f\text{-closed}\big\} \subset C.$$

Lemma 2.2. *Let X be monotonically normal, $S \subset [X]^\omega$ stationary, $x_M \in \overline{M} - M$ for each $M \in S$ and let U_M be an open neighbourhood of the point x_M for $M \in S$. Then there are two sets $M, N \in S$ such that $M \cup \{x_M\} \subset N$ and $x_M \in U_N$.*

Proof. Assume not and put $M < N$ if $M, N \in S$, $M \cup \{x_M\} \subset N$. Choose an open set V_M with $x_M \in V_M \subset \overline{V_M} \subset U_M$ for any $M \in S$ and let V'_M denote the open set assigned to the pair (x_M, V_M) by monotone normality. For each $M \in S$ select a point $x \in M \cap V'_M$. Then $M \to x$ is a regressive function on S. Hence, by Fodor's theorem, there is a stationary $S_0 \subset S$ such that the selected point is the same for each $M \in S_0$. As $V'_M \cap V'_N \neq \emptyset$ for any two $M, N \in S_0$, the definitions of V'_M and V'_N give that $x_M \in V_N$ or $x_N \in V_M$. Now, if $M < N$ then $x_M \notin U_N$ by the indirect assumption so in this case $x_N \in V_M$.

Choose for $M \in S_0$ a set $M^+ \in S_0$ with $M < M^+$ and put $W_M = V_M - \overline{V_{M^+}}$. As before, there is a stationary $S_1 \subset S_0$ such that if $M, N \in S_1$ and $M < N$ then $x_N \in W_M$. Now, if $M, N \in S_1$ with $M < M^+ < N$ then $x_N \in W_M$, $x_N \in V_{M^+}$ and $W_M \cap V_{M^+} = \emptyset$, a contradiction. □

Theorem 2.3. *The following conditions are equivalent for a monotonically normal space X with $t(X) = \omega$*

(a) *X is weakly left separated.*

(b) *The family of the closed countable subsets contains a club in $[X]^\omega$.*

(c) *Any subspace $Y \subset X$ is left separated in type $|Y|$.*

Proof. (c) ⇒ (a) is evident.

(a) ⇒ (b) Fix a neighbourhood-assignment $U(x)$ $(x \in X)$ witnessing that X is weakly left separated. As the countable subsets clearly form a club in $[X]^\omega$, it is enough to prove that $S = \{M \in [X]^\omega\colon M \text{ is not closed in } X\}$ is not stationary in $[X]^\omega$. Assume that it is stationary, select a point $x_M \in \overline{M} - M$ for $M \in S$.

Fix an $M \in S$ and try to choose a sequence $\langle x_k\colon k \in \omega\rangle$ with the following properties: $x_i \in U(x_M) \cap M$, $x_i \in U(x_j)$ for $i < j < \omega$. Our attempt will not succeed because an infinite sequence would contradict to the weak left separatedness of X. The process breaks down at a given point for each $M \in S$: we can construct the sequence $\langle x_i\colon i < k\rangle$ but a suitable x_k does not exist. (Of course, the number k and the sequence $\langle x_i\colon i<k\rangle$ depends on the set $M\in S$.) Take the regressive function $M\to\langle x_i\colon i<k\rangle$. By Fodor's theorem, there is a stationary set $T\subset S$ such that the number k and the assigned sequence is the same for each $M\in T$.

By Lemma 2.2, there are two sets $M,N\in T$, $M<N$ and $x_M\in U(x_N)$. However, this is impossible: we could extend the sequence $\langle x_i\colon i<k\rangle$ with the point $x_k=x_M$ in N.

(b) $\Rightarrow$ (c) It is enough to prove that (b) implies: X is left separated in type $|X|$. Apply the lemma of Kueker to obtain the function $f:[X]^{<\omega}\to X$. Then, by (b), every countable f-closed set is closed. Observe now that any f-closed set $Y\subset X$ is closed. Indeed, any countable subset of Y can be enlarged to an f-closed countable subset of Y hence Y is ω-closed: the closure of any countable subset is contained in Y. As the tightness of X is countable, this shows that Y is closed.

We prove now (c) by transfinite induction on $|X|$, the cardinality of X. A countable T_1-space X is left separated in type ω. Assume now that any subset $Y\subset X$ of size $<|X|$ is left separated (note that the property (b)—just as (a) and (c)—is hereditary). Well-order $X=\{x_\alpha\colon \alpha<\kappa=|X|\}$ and define, by induction, a continuously increasing sequence of f-closed sets F_α such that $\{x_\xi\colon \xi<\alpha\}\subset F_\alpha$, $|F_\alpha|<\kappa$ for $\alpha<\kappa$. Then F_α is closed and left separated. Choose a well-order $\prec_\alpha$ on F_α making the F_α subspace left separated. Put $r(x)=\min\{\alpha\colon x\in F_\alpha\}$ for $x\in X$ and let

$$x\prec y \iff r(x)<r(y) \text{ or } r(x)=r(y)=\alpha \text{ and } x\prec_\alpha y$$

for $x,y\in X$. It is immediate that $\prec$ left separates X. $\square$

The property (b) appears in Fleissner's paper [4, Theorem 2.2]; it is shown there that a space X with a point-countable base is left separated iff it fulfills (b). The proof (b) $\Rightarrow$ (c) is practically taken from [4]. The other conditions mentioned in his theorem, however, are not equivalent for monotonically normal spaces.

We can now prove the main theorem of the paper, Theorem 1.2.

Proof. Using Theorem 2.3, the family S of the not closed countable subsets of X is stationary in $[X]^\omega$. Let $x_M\in\overline{M}-M$ for $M\in S$ and let $c(\vec{x})\in n$ denote the colour assigned to the r-tuple $\vec{x}$ of X. Let us take finally a neighbourhood-base $\{U_i(x)\colon i\in\omega\}$ for any point $x\in X$.

Fix a set $M\in S$ and try to choose a sequence $\langle x_k\colon k\in\omega\rangle$ with the following properties: $x_i\in U_i(x_M)\cap M$, $x_i\neq x_j$ for $i<j<\omega$ and the sequence is *prehomogeneous*: if $0\leqslant j_0<j_1<\dots<j_{r-1}<\omega$ then

$$c\bigl(\langle x_{j_0},x_{j_1},\dots,x_{j_{r-2}},x_{j_{r-1}}\rangle\bigr)=c\bigl(\langle x_{j_0},x_{j_1},\dots,x_{j_{r-2}},x_M\rangle\bigr).$$

Now there are two possibilities.

Case (a). There is an infinite sequence $\langle x_i\colon i<\omega\rangle$ for some $M\in S$. Let $A=\{x_i\colon i<\omega\}$ and colour the $(r-1)$-tuples of A by the rule: $d(\vec{a})=c(\langle\vec{a},x_M\rangle)$. Using the classical Ramsey-theorem $\omega\to(\omega)^{r-1}_n$ we get a d-homogeneous infinite subset $B\subset A$. Now $B\cup\{x_M\}$ is a c-homogeneous convergent sequence.

Case (b). The process breaks down at a given point for each $M \in S$: we can construct the sequence $\langle x_i\colon\ i < k\rangle$ but a suitable x_k does not exist. (Of course, the number k and the sequence $\langle x_i\colon\ i < k\rangle$ depends on the set M.) Take the regressive function $M \to \langle x_i\colon\ i < k\rangle$ on S. By Fodor's theorem, there is a stationary set $T \subset S$ such that the number k and the assigned sequence is the same for each $M \in T$. We can also suppose that the colour $c(\langle x_{j_0}, x_{j_1}, \ldots, x_{j_{r-2}}, x_M\rangle)$ is independent from the choice of the set $M \in T$ for any $0 \leqslant j_0 < j_1 < \cdots < j_{r-2} < k$ (there are only finitely many possibilities).

As no sequence can be continued, we get that if $M, N \in T,\ M < N$ then $x_M \notin U_k(N)$, and this contradicts the Lemma. □

3. An alternative proof

Assume X is a monotonically normal, first countable, not left separated space.

Step 1. Put $\kappa = \min\{|Y|\colon\ Y \subset X, Y$ is not left separated$\}$.

Claim. *κ is regular and uncountable.*

This is a consequence of a result of W. Fleissner. As the conditions given in [4] are different (and there is a serious misprint in the statement of the Theorem), we reproduce here his proof.

Call a space T *moderate* if for any $S \subset T$ $|S| = |\overline{S}|$.

Theorem (Fleissner [4, Theorem 6.1]). *Let T be moderate, $t(T) = \omega$, $|T| = \kappa$ singular and assume that any subspace of T of size less than κ is left separated. Then T is left separated in type $|T|$.*

Proof. We shall prove that T is the union of a continuous chain of small (of size $< \kappa$) closed subsets. This shows the proposition: if $\{F_\alpha\colon\ \alpha < \operatorname{cf}\kappa\}$ is a continuous chain of closed sets, $|F_\alpha| < \kappa$ for $\alpha < \operatorname{cf}\kappa$ and $\bigcup\{F_\alpha\colon\ \alpha < \operatorname{cf}\kappa\} = T$ then choose a well-order $\prec_\alpha$ on F_α witnessing that F_α is left separated for $\alpha < \operatorname{cf}\kappa$. Let $r(x)$ be the minimal α with $x \in F_\alpha$. Now the relation

$$x \prec y \iff r(x) < r(y) \text{ or } r(x) = r(y) = \alpha \text{ and } x \prec_\alpha y$$

is a well-order of type $|T|$ on T and makes T left separated.

Choose a continuously increasing sequence of cardinals $\{\kappa_\alpha\colon\ \alpha < \operatorname{cf}\kappa\}$ with $\operatorname{cf}\kappa < \kappa_0$, $\sup\kappa_\alpha = \kappa$. We shall now define a sequence of length ω_1 of continuous chains $\mathcal{C}^\beta = \{Y_\alpha^\beta\colon\ \alpha < \operatorname{cf}\kappa\}$ ($\beta < \omega_1$) such that

(1) $|Y_\alpha^\beta| = \kappa_\alpha$ for $\beta < \omega_1,\ \alpha < \operatorname{cf}\kappa$;

(2) $\bigcup\{Y_\alpha^\beta\colon\ \alpha < \operatorname{cf}\kappa\} = T$.

Let $\mathcal{C}^0$ be any chain fulfilling (1) and (2). For limit β let $\mathcal{C}^\beta$ be the union.

Assume now that $\beta < \omega_1$ and $\mathcal{C}^\beta$ has been defined. For $\alpha < \operatorname{cf}\kappa$ let $\overline{Y}_\alpha^\beta = \{x_\xi^\alpha\colon \xi < \kappa_\alpha\}$ and put

$$Y_\alpha^{\beta+1} = \{x_\xi^\delta\colon \delta < \operatorname{cf}\kappa,\ \xi < \kappa_\alpha\}.$$

Then $\mathcal{C}^{\omega_1}$ is a continuous chain of closed subsets. In the proof use the (simple and well-known) fact that $t(X) = \omega$ implies that the union of an increasing sequence of ω_1 closed subsets is again closed. □

Proof (*of the claim*). Assume X is monotonically normal, $t(X) = \omega$, $|X| = \kappa$ is singular and any subspace $Y \subset X$ of size $< \kappa$ is left separated; we prove that X is left separated. By Fleissner's theorem, it is enough to prove that X is moderate. If $S \subset X$, $|S| < |\overline{S}|$ then choose a set Y with $S \subset Y \subset \overline{S}$ with $|Y| = |S|^+$. Now Y is not left separated because otherwise the first $|Y|$ element of the well-order would have density $|S|$ by Theorem 1.4 and this is impossible: the density of a space left separated in type λ^+ is at least λ^+. □

Step 2. Assume that $|X| = \kappa$, $\kappa > \omega$ and regular, any subset of size $< \kappa$ is left separated but X is not left separated. Choose a well-order $\prec$ of type κ on X and put (0 denotes the first point in X)

$$S = \{x \in X\colon [0, x) \text{ is not closed in } X\}.$$

Claim. *S is stationary in κ.*

Proof. Assume not, and choose a club C in κ with $C \cap S = \emptyset$ (we identify the sets X and κ). For a $p \in C$ let q be the first point of C greater than p and put $I(p) = [p, q)$. As $I(p)$ is left separated, there exist a well-order $\prec_p$ on $I(p)$ which makes $I(p)$ left separated. For any $x \in X$ let $r(x)$ denote the unique point $p \in C$ with $x \in I(p)$. Put finally

$$x \ll y \iff r(x) < r(y) \text{ or } r(x) = r(y) = p \text{ and } x \prec_p y.$$

Observe that $\ll$ is a well-order on X of type κ and $(X, \ll)$ is left separated. □

Proposition. *If X and κ are as before then there is a not left separated subset $Y \subset X$ and a well-order of type κ on Y such that the set $\{x \in Y\colon x \in \overline{[0, x)}\}$ is stationary in κ.*

Proof. We distinguish two cases.

Case (a). X is not moderate: there is a subset $T \subset X$ with $|T| < |\overline{T}|$. As we have seen in step 1, in this case $Y = \overline{T}$ is not left separated. Now any well-order of type $\kappa = |Y|$ on Y will do the job: the set under discussion contains a final segment.

Case (b). X is moderate. Choose a well-order $\prec$ of type κ on X and put

$$C = \{x \in X\colon \text{for any } y \prec x\ \overline{[0, y)} \subset [0, x)\}$$

(the closure is understood in the topology of X). Evidently C is a club in κ. If S is the stationary set of the claim, the set $T = S \cap C$ is again stationary.

Let now $p \in T$. Then—by $p \in S$—the interval $[0, p)$ is not closed, choose a point $x(p) \in \overline{[0, p)} - [0, p)$. By $T \subset C$, if $p \prec q$, $p, q \in T$ then $x(p) < x(q)$. That means that if $x(p) \neq p$ then $x(p)$ is in the complementary interval with left end-point p. So we can simultaneously replace each p with the corresponding $x(p)$. $\square$

Theorem 3.1. *Let X be a monotonically normal space, $\prec$ a well-order of type $|X| = \kappa$ on X, $\kappa > \omega$ a regular cardinal. If $S \subset X$ is stationary in κ and $x \in \overline{[0, x)}$ for each $x \in S$ then S is not left separated.*

Proof. First we show that S is not discrete. Assume, on the contrary, that there exists an open neighbourhood U_x for each $x \in S$ with $U_x \cap S = \{x\}$. Let U'_x denote the open set assigned to the pair (x, U_x) by the definition of monotone normality. Then the sets U'_x $(x \in S)$ are pairwise disjoint. However, by $x \in \overline{[0, x)}$ there would be a point $y(x) \in U'_x$, $y(x) \prec x$ for each $x \in S$. Now $x \to y(x)$ is a regressive function on S, hence, by Fodor's theorem, there are different x's with identical $y(x)$, a contradiction.

Assume now that S is left separated, i.e., there is a well-order on S which makes S left separated. By Lemma 1.5 the two well-orders are identical on a stationary subset of S, so we can suppose that S is left separated by the given well-order $\prec$. There exists an open neighbourhood U_x for each $x \in S$ with $x \notin \overline{U_y}$ if $x, y \in S$, $x \prec y$. Let, as before, U'_x denote the open x-neighbourhood corresponding to the pair (x, U_x) by monotone normality. Using again the Fodor's theorem, we get a stationary set $T \subset S$ with $\bigcap\{U'_x\colon x \in T\} \neq \emptyset$. Now, by the definition of U'_x, we get that $y \in U_x$ if $x, y \in T$, $x \prec y$. For each $x \in T$ denote by x^+ the first point of T greater than x and put $V_x = U_x - \overline{U_{x^+}}$. The neighbourhoods V_x show that T is discrete, a contradiction. $\square$

The following theorem sums up the results of Steps 1 and 2.

Theorem 3.2. *Let X be monotonically normal, not left separated, $|X| = \kappa$ and assume that any subspace of X of size $< \kappa$ is left separated. Then κ is regular and there is a well-order of type κ on X and a stationary subset $S \subset X$ such that any stationary subset of S is not left separated.*

Step 3. Now we can prove Theorem 1.2, the main theorem of the paper.

Proof. By steps 1 and 2 we can suppose that X fulfills the condition of Theorem 2.2: $|X| = \kappa$ is a regular cardinal, there is a well-order $\prec$ of type κ on X and there is a stationary subset $S \subset X$ such that any stationary subset of S is not left separated. Fix a neighbourhood-base $\{U_k(x)\colon k \in \omega\}$ for any $x \in X$. Let $c(\vec{x}) \in n$ denote the colour assigned to the r-tuple $\vec{x} = \langle x_0, x_1, \ldots, x_{r-1}\rangle$, $x_0 \prec x_1 \prec \cdots \prec x_{r-1}$, $x_0, x_1, \ldots, x_{r-1} \in X$.

Fix a point $x \in S$ and try to choose a sequence $\langle x_k\colon k \in \omega\rangle$ with the following properties: $x_i \in U_i(x)$, $x_i \prec x_j \prec x$ for $i < j < \omega$ and the sequence is *prehomogeneous*: if $0 \leqslant j_0 < j_1 < \cdots < j_{r-1} < \omega$ then

$$c\bigl(\langle x_{j_0}, x_{j_1}, \ldots, x_{j_{r-2}}, x_{j_{r-1}}\rangle\bigr) = c\bigl(\langle x_{j_0}, x_{j_1}, \ldots, x_{j_{r-2}}, x\rangle\bigr).$$

Now there are two possibilities.

Case (a). There is an infinite sequence $\langle x_i\colon i<\omega\rangle$ for some $x\in S$. Let $A=\{x_i\colon i<\omega\}$ and colour the $(r-1)$-tuples of A by the rule: $d(\vec{a})=c(\langle\vec{a},x\rangle)$. Using the classical Ramsey-theorem $\omega\to(\omega)^{r-1}_n$ we get a d-homogeneous infinite subset $B\subset A$. Now $B\cup\{x\}$ is a c-homogeneous convergent sequence.

Case (b). For each $x\in S$ the process breaks down at a given point: we can construct the sequence $\langle x_i\colon i<k\rangle$ but a suitable x_k does not exist. (Of course, the number k and the sequence $\langle x_i\colon i<k\rangle$ depends on the point x.) Take the regressive function $x\to\langle x_i\colon i<k\rangle$. By Fodor's theorem, there is a stationary set $T\subset S$ such that the number k and the assigned sequence is the same for each $x\in T$. We can also suppose that the colour $c(\langle x_{j_0},x_{j_1},\dots,x_{j_{r-2}},x\rangle)$ is independent from the choice of the point $x\in T$ for any $0\leqslant j_0<j_1<\cdots<j_{r-2}<k$ (there are only finitely many possibilities). As no sequence can be continued, we get that if $x,y\in T$, $x\prec y$ then $x\notin U_k(y)$, i.e., T is left separated, a contradiction. $\square$

4. Examples and remarks

Hence it is a natural question: is the converse to Theorem 1.1 true in the class of the (say) Tychonoff spaces? The following example shows that this is not necessarily so.

Example 4.1 (*not* CH). A not left separated space with a colouring of the pairs such that any homogeneous subset is discrete.

Let X be the set of ω_1-limit ordinals in ω_2. Then X is not left separated by Theorem 3.1. If the continuum hypothesis is false then $|X|=\omega_2\leqslant 2^\omega$ and so the Sierpinski-colouring on X enables only countable homogeneous subsets which are discrete (and closed) in X.

We do not have a first countable example nor any such example in ZFC.

With the continuum hypothesis the situation radically changes. Indeed, CH implies that *colouring the pairs of* X of Example 4.1 *with finitely many colours there exists a not discrete homogeneous subset*.

Proof. Let $c:[X]^2\to 2$ be a colouring of the pairs of a stationary subset $S\subset X$. Fix an $\alpha\in S$ and a sequence $\langle\alpha_\xi\colon\xi<\omega_1\rangle$ of ordinals increasingly converging to α. Try to choose a sequence $\langle x_\xi\colon\xi<\omega_1\rangle$ with $\alpha_\xi<x_\xi<x_\eta<\alpha$, $c(x_\xi,x_\eta)=c(x_\xi,\alpha)=0$ for $\xi<\eta<\omega_1$. If there is an $\alpha\in S$ with such a sequence, we get a not discrete 0-homogeneous set. So we can suppose that each sequence breaks down at a countable ordinal. By Fodor's theorem, there is a stationary set $S_0\subset S$ and an ordinal ξ such that for any $\alpha\in S_0$ the sequence assigned to α breaks down under ξ. By CH, there are only $\omega_1<\omega_2$ possible sequences under ξ, so there is a stationary $S_1\subset S_0$ with the same sequence. Now the colour of any pair in S_1 is 1.

Summing up: colouring the pairs of a stationary set $S\subset X$ with 2 colours, either there is a not discrete 0-homogeneous set or there is a 1-homogeneous stationary $T\subset S$. The theorem now follows by induction. $\square$

Moreover: as $2^{2^\omega} \not\to (\omega_1)^3_2$ (see, e.g., [3]) and any countable subset of X is discrete, we get that, in ZFC, we can colour the triples of X with 2 colours such that any homogeneous subset is discrete.

This space is monotonically normal, fulfills the condition (b) of Theorem 2.3 (any countable subset is closed hence $[X]^\omega$ is a suitable club in $[X]^\omega$), but it is not left separated.

We now give an example for a weakly left separated but not left separated space. Let us begin with a lemma.

Lemma 4.2. *If X is regular, dense in itself, Baire and separable then it is not left separated.*

Proof. Assume the contrary and take the shortest counterexample X. As any not empty open subset is again a counterexample, any not empty open set is cofinal. We get that each final segment of X is a dense open set. The separability of X implies that the cofinality of the given left ordering is ω. Consequently the intersection of countably many dense open subsets of X is empty, X is not Baire. □

Example 4.3. There is a Tychonoff space X with the following properties: X is not left separated, it is the union of a left separated and of a countable space, $|X| = 2^\omega$ and any subset of size less than continuum is left separated.

Construction. Take $T = D(2)^{2^\omega}$. Let Φ be the set of the partial functions defined on a countable subset of 2^ω with values 0 and 1. For $\phi \in \Phi$ put $B(\phi) = \{f \in T\colon \phi \subset f\}$. Let $\mathcal{B} = \{B(\phi)\colon \phi \in \Phi\} = \{B_\xi\colon \xi < 2^\omega\}$. Evidently $\mathcal{B}$ is a base of the G_δ-topology on T. For a set $B = B(\phi)$ put $I(B) = \operatorname{dom}\phi$.

We now construct a sequence $\{f_\xi\colon \xi < 2^\omega\}$ such that $f_\xi \in B_\xi \cap \Sigma$ for $\xi < 2^\omega$ and $S = \{f_\xi\colon \xi < 2^\omega\}$ is left separated. (Σ, as usual, denotes the set of the functions vanishing outside a countable set.) Assume that $\alpha < 2^\omega$ and f_ξ has been defined for $\xi < \alpha$. Put $I_\xi = f_\xi^{-1}(1)$. Choose now an $i \in 2^\omega - (I(B_\alpha) \bigcup \{I_\xi\colon \xi < \alpha\})$ and a point $f \in B_\alpha \cap \Sigma$ with $f(i) = 1$. The set $\{f_\xi\colon \xi < 2^\omega\}$ is left separated and meets any not empty G_δ-set of T. As T is separable, there is a countable dense set C in T. It is easily seen that we can also suppose: any point in C has continuum many 1 coordinates.

Now $X = S \cup C$ is not left separated by the previous Lemma. If $R \subset S$ has cardinality less than continuum, then $\overline{R} \cap C = \emptyset$ hence $R \cup C$ is left separated (put C before R in the well-order).

The space is weakly left separated as the union of finitely many (weakly) left separated subspaces, but does not fulfill the condition (b) in Theorem 2.3. If the continuum is a singular cardinal, then this example shows that Fleissner's theorem is not true without any assumption on the space X.

Question. Is a monotonically normal, weakly left separated space left separated? In other words: can we omit the condition $t(X) = \omega$ from (a) $\Rightarrow$ (c) in Theorem 2.5?

The fact that a weakly left separated and monotonically normal space X with tightness ω is left separated can be proved by the method of Section 3, too. Take a counterexample of minimal cardinality and apply the argument of Section 3 to the space. (The details are left to the reader.) The condition $t(X)=\omega$ is used only in Fleissner's theorem, when we show that $|X|=\kappa$ is a regular cardinal. However, this is not necessary if $|X|<\aleph_{\omega_1}$ because $\operatorname{cf}\kappa=\omega$ is impossible: the union of countably many closed left separated subspace is left separated. Hence a monotonically normal, weakly left separated space X with $|X|<\aleph_{\omega_1}$ is left separated.

A simple example for a left separated space X which is not left separated in type $|X|$: take a separable, uncountable left separated space (see, e.g., [2, Example 3.6.I(a)]).

Another example: Let $X=\omega_1$, I the set of isolated points, L the set of limit-ordinals.

For each $\xi\in L$ choose a set $S_\xi\subset\xi\cap I$ of type ω converging to the point ξ. Let any point in I be isolated, and a neighbourhood of an ordinal $\xi\in L$ let be the family of the sets of the form $(S_\xi-V)\cup\{\xi\}$, where V is a finite set. It is not hard to see that the least type in which X can be left separated, is $\omega_1+\omega_1$.

The next result also uses the machinery developed in Section 3.

Theorem 4.4. *If X is monotonically normal, not left separated, $t(X)=\omega$ and X is the union of κ many left separated subspace then X contains a not left separated subspace of size less or equal to κ.*

Proof. Let $X=\bigcup\{X_\alpha\colon \alpha<\kappa\}$, where the subspaces X_α are left separated. Apply the argument of Section 3 to X. We get a subspace $Y\subset X$, a regular cardinal $\lambda=|Y|$, a well-order of Y in type λ and a stationary subset $S\subset\lambda$ such that no stationary subset of S is left separated. Now, if $\lambda\leqslant\kappa$ then S (or Y) is an appropriate subspace. If $\kappa<\lambda$ then at least one of the subspaces $S_\alpha=S\cap X_\alpha$ would be stationary. However, this is impossible, because S_α can not be left separated. $\square$

Corollary 4.5. *If a monotonically normal space with countable tightness is the union of countably many left separated subspaces, then it is left separated.*

The last theorem of this paper again uses the characterization of the left separated spaces proved in Section 3. By Fleissner's theorem, the minimal cardinality of a not left separated subspace of a metrizable space is always regular. Can we assert more? Is it true, e.g., that if a metrizable space is not left separated, then it contains a not left separated subspace of size ω_1? Fleissner proved [4] that the answer is independent from ZFC. The next theorem gives a precise characterization. Let us begin with a simple result about metric spaces.

Call a subset R of a metric space (X,d) *rare* if there is an $\varepsilon>0$ such that for any two points $x,y\in R$, $x\neq y$, $d(x,y)\geqslant\varepsilon$ holds.

Lemma 4.6. *The following conditions are equivalent for a metric space X:*

(a) *X is left separated.*
(b) *X is σ-discrete.*
(c) *X is σ-rare.*

Proof. (c) $\Rightarrow$ (b) is evident, (b) $\Rightarrow$ (a) by Corollary 4.5. To show (a) $\Rightarrow$ (c) let X be left separated and let $\prec$ and $x \to U(x)$ $(x \in X)$ witness this. Put $R_n = \{x \in X\colon\ S(x, 1/n) \subset U(x)\}$. Then R_n is rare and $\bigcup\{R_n\colon\ n \in \omega\} = X$. $\square$

Remark. A left separated, not σ-discrete monotonically normal first countable space is given in [4, Example B].

Theorem 4.7. *Let $\kappa > \omega$ be a regular cardinal. The following conditions are equivalent*:

(a) *There is a not left separated metric space X, $|X| = \kappa$, any subset of X of size $< \kappa$ is left separated.*

(b) *There is a stationary subset E of the ω-limits ordinals in κ and a sequence of functions $\{f_n\colon\ n < \omega\}$ with $f_n(\xi) < \xi$, $\sup_n f_n(\xi) = \xi$ for $\xi \in E$ such that for any $\alpha < \kappa$ $E \cap \alpha = \bigcup\{A_n\colon\ n \in \omega\}$ and for each n there is a $k(n) \in \omega$, the restriction of $f_{k(n)}$ to A_n is* 1–1.

Proof. (a) $\Rightarrow$ (b). Apply the argument of Section 3 to the space X. Denote E the stationary subset of κ and let $\{f_n(\xi)\colon\ n \in \omega\}$ be a sequence converging to the ordinal $\xi \in E$. If $\alpha < \kappa$ then the subset $E \cap \alpha$ is left separated, so σ-rare by the preceding Lemma: $E = \bigcup\{R_n\colon\ n \in \omega\}$. For a fixed n, we can choose pairwise disjoint neighbourhoods $U(\xi)$ for the points $\xi \in E_n$. Select an $f_k(\xi) \in U(\xi)$ and put $\xi \in E_{n,k}$ if the chosen index is k. This works.

(b) $\Rightarrow$ (a). Put $P(\xi) = (f_0(\xi), f_1(\xi), f_2(\xi), \ldots) \in M$ and let $M = D(\kappa)^{\omega}$. Then $X = \{P(\xi)\colon\ \xi \in E\}$ is a suitable metric space. $\square$

Remark. It is not hard to see that if $\kappa > \omega$ is regular and E is a stationary subset of the ω-limits in κ with property E (i.e., $E \cap \alpha$ is not stationary in α for each $\alpha \in \kappa$) then the set E fulfills the condition (b). It would be interesting to know, is (b) actually equivalent to the property E?

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